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Generalized Relativistic Fock Space

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Abstract

We consider a generalized Fock space obtained by eliminating the restriction to symmetric components for bosons or antisymmetric ones for fermions. In this space we can extend the many times formalism of relativistic quantum mechanics to quantum field theory, in which each particle has a time parameter that has to be included in any exchange of variables. Physical states in which all particle times, or all antiparticle times, are equal, still have the right symmetry. We define creation and annihilation operators for numbered particles in this space, and relate them to the usual operators.

1. *Introduction*

We previously defined (Marx, 1972) in a somewhat modified manner the creation and annihilation operators in a relativistic Fock space. The changes were suggested by the probabilistic interpretation of relativistic quantum mechanics, especially important in configuration space. We have primarily used normalized functions to describe the states in which particles are created or annihilated, although they can be formally replaced by Dirac δ-functions when this is found to be more convenient.

Our theory of relativistic quantum mechanics is based on Dirac's many times formalism, and pair creation and annihilation are taken into account by reversing the direction of propagation of the wave function in time. We thus obtain a theory with a fixed number of 'particles', which can be found either in a particle state or an antiparticle state. The wave functions in such a theory are either symmetric or antisymmetric under the exchange of *all* variables corresponding to two particles, which in particular includes the time parameters. For instance, for a two-particle boson amplitude we have

$$\psi^{(++)}(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2) = \psi^{(++)}(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1) \tag{1.1}$$

This does *not* imply that the amplitudes for fixed values of t_1 and t_2 are symmetric functions of $\mathbf{x}_1$ and $\mathbf{x}_2$. Only when the times are equal do we obtain the usual symmetry relations, that is, equation (1.1) becomes

$$\psi^{(++)}(\mathbf{x}_1, \mathbf{x}_2; t) = \psi^{(++)}(\mathbf{x}_2, \mathbf{x}_1; t) \tag{1.2}$$

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We note that the observer, implicit in any theory formulated in the Schrödinger picture, selects a preferred time direction in Minkowski space, and the states of the system are described on hyperplanes perpendicular to this direction, by functions of three-vector variables. The 'particle' times are parameters that characterize the motion of the state vector in a dynamical problem.†

We have assumed that we start with a given state in which the particles are all at the initial time and antiparticles all at the final time, and the results of a dynamical calculation are amplitudes with particles at the final time and antiparticles at the initial time. All these amplitudes have the right symmetry. But the time development of the system through equations of motion includes all intermediate times, where this is not the case.‡ Creation operators defined (Schweber, 1961; Marx, 1970a, b, 1972) for a Fock space with symmetric or antisymmetric components have this property built into them, and new operators that only affect a specific variable are needed in the generalized Fock space.

In this paper we consider only the state vectors that describe the systems and not their time development. To avoid unnecessary complications, we assume that only components with a fixed number of variables differ from zero, although many equations apply to a more general state.

We present the new creation and annihilation operators and their anti-commutation relations in the case of spin-$\frac{1}{2}$ fermions in Section 2. In Section 3 we relate these operators to those defined for the more restricted Fock space. We briefly discuss the changes that have to be made for the case of spinless bosons in Section 4, and we conclude with some remarks in Section 5. The notation we use follows closely that of Marx (1972).

2. *Numbered 'Particle' Creation and Annihilation Operators*

We designate the creation and annihilation operators for 'particle' j in a state b by§ $R_j^{(\kappa)}(b)$ and $L_j^{(\kappa)}(b)$ respectively. We define them by giving the components of Ψ' and Ψ'', for an arbitrary state vector Ψ, where

$$\Psi' = R_j^{(\kappa)}(b)\Psi \tag{2.1}$$

$$\Psi'' = L_j^{(\kappa)}(b)\Psi \tag{2.2}$$

† We recover the symmetry properties of the amplitudes if we consider them as vectors in a different Hilbert space, that of functions of several four-vector variables. We prefer not to do this, since we find that it obscures the physical interpretation of the theory and changes the basic definitions of quantum mechanics.

‡ This was overlooked in the case of identical bosons in an electromagnetic field (Marx, 1970b), and the presentation of the dynamics in Fock space has to be changed accordingly.

§ The index κ refers to the mode of propagation, and ranges over plus for particles and minus for antiparticles. The state b is given by two amplitudes normalized to 1, that is, if we use momentum space amplitudes, we have

$$\sum_{\lambda=\pm} \int d^3 k |b_\lambda(\mathbf{k})|^2 = 1$$

They are†

$$\psi'^{(\kappa_1\ldots\kappa_n)}_{(1,\ldots,n)} = (-1)^{j-1}\,\delta^{\kappa\kappa_j}\, b_j\, \psi^{(\kappa_1\ldots\kappa_{j-1}\,\kappa_{j+1}\ldots\kappa_n)}_{(1,\ldots,j-1,j+1,\ldots,n)} \tag{2.3}$$

$$\psi''^{(\kappa_1\ldots\kappa_n)}_{(1,\ldots,n)} = (-1)^{j-1}\, b_i^*\, \psi^{(\kappa_1\ldots\kappa_{j-1}\,\kappa\kappa_j\ldots\kappa_n)}_{(1,\ldots,j-1,i,j,\ldots,n)} \tag{2.4}$$

when these equations are applicable, and zero otherwise. It is now straightforward to show that the creation operator is the Hermitian conjugate of the annihilation operator with respect to the metric

$$(\Psi,\Psi') = \psi^{(0)*}\psi'^{(0)} + (\psi^{(\kappa_1)},\psi'^{(\kappa_1)}) + (\psi^{(\kappa_1\kappa_2)},\psi'^{(\kappa_1\kappa_2)}) + \ldots \tag{2.5}$$

Instead of the usual anticommutation relation, we find that

$$R_j^{(\kappa)}(b)\,R_{j'}^{(\kappa')}(b') + R_{j'+1}^{(\kappa')}(b')\,R_j^{(\kappa)}(b) = 0, \qquad j \leqslant j' \tag{2.6}$$

$$L_j^{(\kappa)}(b)\,L_{j'}^{(\kappa')}(b') + L_{j'}^{(\kappa')}(b')\,L_{j+1}^{(\kappa)}(b) = 0, \qquad j \geqslant j' \tag{2.7}$$

$$L_j^{(\kappa)}(b)\,R_{j'}^{(\kappa')}(b') + R_{j'}^{(\kappa')}(b')\,L_{j-1}^{(\kappa)}(b) = 0, \qquad j > j' \tag{2.8}$$

$$L_j^{(\kappa)}(b)\,R_{j'}^{(\kappa')}(b') + R_{j'-1}^{(\kappa')}(b')\,L_j^{(\kappa)}(b) = 0, \qquad j < j' \tag{2.9}$$

$$L_j^{(\kappa)}(b)\,R_{j'}^{(\kappa')}(b') = (b,b')\,\delta^{\kappa\kappa'}, \qquad j = j' \tag{2.10}$$

We note that Ψ' is normalized when Ψ has norm 1, but that this is not necessarily the case for Ψ''.

As we have done before, we can extend the definitions to include the Dirac δ-function and set in the place of b

$$b_{\lambda\mathbf{k}} = \delta_{\lambda\lambda_j}\,\delta(\mathbf{k}-\mathbf{k}_j) \tag{2.11}$$

to define $R_{j\lambda}^{(\kappa)}(\mathbf{k})$ and $L_{j\lambda}^{(\kappa)}(\mathbf{k})$.

3. *Restriction to Antisymmetric Amplitudes*

When the usual assumption that all the times are equal is made, we recover the condition that all amplitudes have certain symmetry conditions. The usual creation and annihilation operators are defined within this subspace, and are related to the operators defined above by‡

$$R^{(\kappa)}(b) = n^{-1/2}\sum_{j=1}^{n} R_j^{(\kappa)}(b) \tag{3.1}$$

$$L^{(\kappa)}(b) = (n+1)^{-1/2}\sum_{j=1}^{n+1} L_j^{(\kappa)}(b) \tag{3.2}$$

We note that we obtain $n+1$ equal terms from the sum in equation (3.2), which gives the normalization factor $(n+1)^{1/2}$ found previously (not that in

† A subindex on a function stands for both the continuous variable and spin index. When such an index is repeated, it implies an integration over the former and a summation over the latter.

‡ Here we assume that the state vectors Ψ' and Ψ'' have only components for n 'particles'.

Schweber, where a different scalar product in Fock space is used). Neither Ψ' nor Ψ'' are necessarily normalized now.

With some attention paid to the number of particles in the different states and the restrictions on the indices in equations (2.6) to (2.10), we can easily show† that the operators defined by equations (3.1) and (3.2) obey the usual anticommutation relations.

4. *Identical Bosons*

We only have to introduce a few changes in the equations of the last two sections to apply them to the case of scalar bosons. There is, of course, no longer a spin index and b stands for a single function. The phase factor $(-1)^{j-1}$ in equations (2.3) and (2.4) leads to anticommutation relations, and we eliminate it to define the boson operators; the new states are given by

$$\psi'^{(\kappa_1\ldots\kappa_n)}_{(1,\ldots,n)} = \delta^{\kappa\kappa_j} b_j \psi^{(\kappa_1\ldots\kappa_{j-1}\kappa_{j+1}\ldots\kappa_n)}_{(1,\ldots,j-1,j+1,\ldots,n)} \tag{4.1}$$

$$\psi''^{(\kappa_1\ldots\kappa_n)}_{(1,\ldots,n)} = b_i^* \psi^{(\kappa_1\ldots\kappa_{j-1}\kappa\kappa_j\ldots\kappa_n)}_{(1,\ldots,j-1,i,j,\ldots,n)} \tag{4.2}$$

Creation and annihilation operators now obey commutation relations obtained by changing the plus signs in equations (2.6) to (2.9) to minus signs.

We can use equations (3.1) and (3.2) as they stand for equal-time amplitudes, which in this case are symmetric. The resulting operators obey the usual commutation relations.

5. *Concluding Remarks*

We have defined creation and annihilation operators in a generalized Fock space, in which the states are not restricted to symmetric or antisymmetric amplitudes. The operators in this Fock space can be used to formulate a second quantized version of the many times formalism of relativistic quantum mechanics. Such a theory deals with a fixed number

† For instance, if we assume that the operators apply to a state Ψ with n 'particles', we have

$$\begin{aligned}
(n+1)^{1/2}(n+2)^{1/2} R^{(\kappa)}(b) R^{(\kappa')}(b') &= \left[\sum_{j=1}^{n+2} R_j^{(\kappa)}(b)\right]\left[\sum_{j'=1}^{n+1} R_{j'}^{(\kappa')}(b')\right] \\
&= \sum_{j=2}^{n+2}\sum_{j'=1}^{j-1} R_j^{(\kappa)}(b) R_{j'}^{(\kappa')}(b') + \sum_{j=1}^{n+1}\sum_{j'=j}^{n+1} R_j^{(\kappa)}(b) R_{j'}^{(\kappa')}(b') \\
&= -\sum_{j=2}^{n+2}\sum_{j'=1}^{j-1} R_{j'}^{(\kappa')}(b') R_{j-1}^{(\kappa)}(b) - \sum_{j=1}^{n+1}\sum_{j'=j}^{n+1} R_{j'+1}^{(\kappa)}(b') R_j^{(\kappa)}(b) \\
&= -\sum_{j=1}^{n+1}\sum_{j'=1}^{j} R_{j'}^{(\kappa')}(b') R_j^{(\kappa)}(b) \\
&\quad - \sum_{j=1}^{n+1}\sum_{j'=j+1}^{n+2} R_{j'}^{(\kappa')}(b') R_j^{(\kappa)}(b) \\
&= -(n+1)^{1/2}(n+2)^{1/2} R^{(\kappa')}(b') R^{(\kappa)}(b)
\end{aligned}$$

of 'particles', and we have used this property to relate our operators to the usual ones, but this is not a basic restriction in either case.

The extension to particles with higher spin or other multiplicities such as isospin is straightforward, and we can also extend the definitions to allow for different types of particles (Marx, 1972).

Physical states in which all particle or antiparticle times are equal still have the usual symmetry properties, but states at intermediate times do not. Hence the need to change the Fock space in order to formulate the dynamics of such a theory.

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